INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 8, 91–96 (1988)

NUMERICAL SOLUTION OF A FREE SURFACE PROBLEM BY A BOUNDARY ELEMENT METHOD

JOYCE M. AITCHISON

Mathematics and Ballistics Group, Royal Military College of Science, Shrivenham, Swindon, Wilts SN6 8LA, U.K.

AND

ANDREAS KARAGEORGHIS

Department of Mathematics, The University College of Wales, Aberystwyth, Dyfed SY23 3BZ, U.K.

SUMMARY

This paper describes a method for the numerical solution of a Riabouchinsky cavity flow. Application of a boundary element method leads to a system of non-linear equations. The mild singularity appearing at the separation point is treated with the introduction of a curved boundary element, which satisfies the exact behaviour of the free boundary in that neighbourhood.

KEY WORDS Boundary elements Free surface Riabouchinsky cavity flow

INTRODUCTION

This paper considers the cavitational flow of an incompressible inviscid fluid past a plate in a channel of finite width and infinite length. The plate is placed symmetrically in the channel at right angles to the flow. When the flow meets the plate, a cavity is formed immediately behind the plate, containing air at a constant pressure.

Various theoretical models have been suggested for this problem and a review of these can be found in Wu[1] or Birkhoff and Zarantello.[2] The most popular model available was put forward by Riabouchinsky.[3] In it, the cavity is closed by introducing an image plate downstream of the plate, as shown in Figure 1. As the flow is also symmetric about the axis of the channel, only a quarter of the region needs to be considered.

The model described above can be used for the solution of two different problems, a planar flow and an axisymmetric flow. Numerical solutions for the planar flow problem were obtained by Mogel and Street,[4] using a finite difference method, and by Aitchison,[5] using the method of variable finite elements. The axisymmetric flow problem has also received much attention (see Aitchison,[5] Brennen[6] and Fox and Sankar[7]). This study concentrates on the planar flow problem.

The model can be made non-dimensional (for details, see, Aitchison[5]) and this leads to the following problem for the streamfunction ψ in the region Ω (Figure 2):

$$\nabla^2\psi = 0 \quad \text{in } \Omega, \tag{1}$$

$$\partial\psi/\partial n = 0 \quad \text{on AB, CD}, \tag{2}$$

$$\psi = 0 \quad \text{on DE, EO}, \tag{3}$$

0271–2091/88/010091–6$05.00

Received 1 December 1986
Revised 23 March 1987

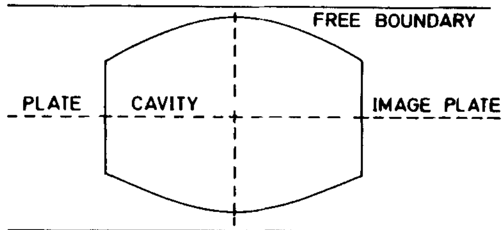

Figure 1. Riabouchinsky's model

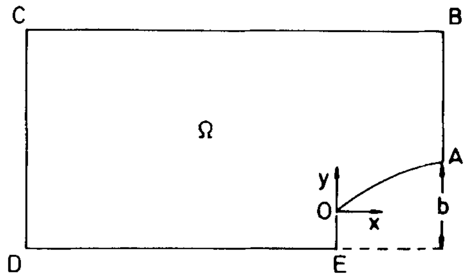

Figure 2. Region of solution

$$\psi = 1 \quad \text{on BC} \tag{4}$$

and

$$\left.\begin{array}{l}\psi = 0\\ \partial\psi/\partial n = q_c\end{array}\right\} \quad \text{on the free boundary OA,} \tag{5}$$

where $\partial/\partial n$ denotes the derivative in the direction of the outward normal to the boundary of Ω and q_c is a constant to be determined as part of the solution.

The fact that the value of q_c is not known makes the problem particularly difficult. Advantage, however, can be taken of the fact that q_c is a constant. If the boundary condition $\psi = 0$ is assumed to hold on the free boundary OA, the latter can, in principle, be iterated until a position is reached at which $\partial\psi/\partial n$ is constant on it.

It should also be mentioned that there is a mild singularity at the separation point which is considered later in the numerical solution of the problem.

APPLICATION OF THE DIRECT BOUNDARY ELEMENT METHOD

The most interesting feature of the problem described in the 'Introduction' is the existence of a free boundary. Boundary element methods deal directly with the boundary of the region under consideration and are therefore very suitable for the numerical solution of such problems, as has been demonstrated by Liggett,[8] Kelmanson,[9] Aitchison and Karageorghis[10] and Karageorghis.[11]

The direct boundary element method is based on Green's third identity in the form (for details, see, e.g., Jaswon and Symm[12]):

$$\alpha(\mathbf{p})\psi(\mathbf{p}) = \int_{\partial\Omega}\left(\psi(\mathbf{q})\frac{\partial \log|\mathbf{p}-\mathbf{q}|}{\partial n} - \log|\mathbf{p}-\mathbf{q}|\frac{\partial\psi(\mathbf{q})}{\partial n}\right)\mathrm{d}s(\mathbf{q}), \tag{6}$$

where

$$\alpha(\mathbf{p}) = \begin{cases} 0 & \text{if } \mathbf{p} \notin \Omega \cup \partial\Omega, \\ \alpha & \text{if } \mathbf{p} \in \partial\Omega, \\ 2\pi & \text{if } \mathbf{p} \in \Omega, \end{cases} \tag{7}$$

α is equal to π if the boundary $\partial\Omega$ has a unique tangent at $\mathbf{p}$, otherwise it is equal to the interior angle between the tangents to $\partial\Omega$ at $\mathbf{p}$, and $\partial/\partial n$ denotes the derivative in the direction of the outward normal to $\partial\Omega$ at $\mathbf{q}$. The boundary $\partial\Omega$ is divided into N straight line segments. On each such segment (element), the functions ψ and $\partial\psi/\partial n$ are approximated by piecewise linear functions, in terms of their nodal values ; i.e., their values at the endpoints of the segment. Double nodes are used at the corners. Subsequent collocation at each node leads to the system of N equations

$$\mathbf{A}\boldsymbol{\psi} - \mathbf{B}\boldsymbol{\psi}^{(n)} = \mathbf{E}\boldsymbol{\psi}, \tag{8}$$

where $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^{(n)}$ are the vectors of the values of ψ and $\partial\psi/\partial n$ respectively at each node. $\mathbf{E}$ is a diagonal matrix consisting of the values of $\alpha(\mathbf{p})$ at each node. $\mathbf{A}$ and $\mathbf{B}$ are matrices consisting of the weighted integrals of $(\partial/\partial n)\log|\mathbf{p}-\mathbf{q}|$ and $\log|\mathbf{p}-\mathbf{q}|$ respectively over each element. These integrals are evaluated analytically in the manner described in Ingham *et al.*[13]

Given ψ or $\partial\psi/\partial n$ or a linear combination of these at each node, substitution into (8) leads to a system of N linear equations in N unknowns which is solved by Gaussian elimination. If the values of ψ at interior points are required, having obtained the unknowns at the boundary nodes, a second step using (6) evaluates these.

NUMERICAL PROCESS

The obvious choice of variables to represent the free boundary in a suitable manner would be the y co-ordinates of the nodes on it. Iterating each of these separately, however, would be very uneconomical and tedious and therefore out of the question. Instead, the following procedure is adopted:

Consider the direct boundary element method for the solution of the problem and let $\mathbf{h}$ be the vector of values of the y co-ordinates of the nodes on the free boundary.

At each node on the fixed part of the boundary (not on the free boundary), let the unknowns be ψ and $\partial\psi/\partial n$. At each node on the free boundary (excluding the node at O), the unknowns are ψ, $\partial\psi/\partial n$ and the y co-ordinates $\mathbf{h}$.

If there are n nodes on the fixed part of the boundary and m nodes on the free boundary, the total number of unknowns is therefore

$$M = 2n + 3m. \tag{9}$$

The direct boundary element method generates $n + m$ equations (total number of nodes). The boundary conditions for each of the nodes on the fixed part of the boundary provide n equations. On the free boundary, both boundary conditions are imposed:

$$\psi = 0, \tag{10}$$

$$\partial\psi/\partial n = \text{constant}. \tag{11}$$

Condition (10) gives m equations. Condition (11), if written in the form

$$\partial\psi/\partial n|_i = \partial\psi/\partial n|_o, \quad i = 1, 2, \ldots m \tag{12}$$

(i is a node on the free boundary), provides m equations.

The total number of equations is therefore

$$2n + 3m = M; \tag{13}$$

i.e., there is an equal number of equations and unknowns.

Assuming that condition (10) holds on the free boundary and fixing $\mathbf{h}$, the values of $\partial\psi/\partial n$ can be obtained on the free boundary. The values of $\partial\psi/\partial n$ obtained are therefore entirely dependent on the position of the free boundary—i.e., $\mathbf{h}$—and the requirement that $\partial\psi/\partial n$ takes constant values there leads to the solution of the system of non-linear equations

$$f_i(\mathbf{h}) = \partial\psi/\partial n|_i - \partial\psi/\partial n|_o = 0, \quad i = 1, 2, \ldots, m, \tag{14}$$

where i is a node on the free boundary.

It has thus been demonstrated that the determination of the position of the free boundary and the constant q_c is equivalent to the solution of a system of non-linear equations. The system can be solved numerically using a modified Powell-hybrid method, details for which can be found in Powell.[14]

INCLUSION OF A CURVED ELEMENT AT THE SINGULARITY

So far, no account has been taken of the mild singularity at the separation point O. From past work on the problem (see Mogel and Street[4] and Aitchison[5]), the exact nature of the free boundary in the neighbourhood of the singularity is known. When using the direct boundary element method, full advantage can be taken of this, as a curved element can easily be introduced at O, satisfying this known behaviour, while still employing straight line segments to describe the rest of the boundary.

The shape of the free boundary near the singularity is given by

$$y = ax^{2/3}, \tag{15}$$

where a is a constant to be determined. This behaviour is taken on the curved element OF (see Figure 3) and, if the co-ordinates of the node at F are (x_1, y_1), then

$$a = y_1/x_1^{2/3}. \tag{16}$$

Taking the parameter s to describe the arc length from O to F, this is given in terms of x as

$$s(x) = \int_0^x [1 + (dy/dx)^2]^{1/2} dx = (x^{2/3} + \tfrac{4}{9}a^2)^{3/2} - \tfrac{8}{27}a^3. \tag{17}$$

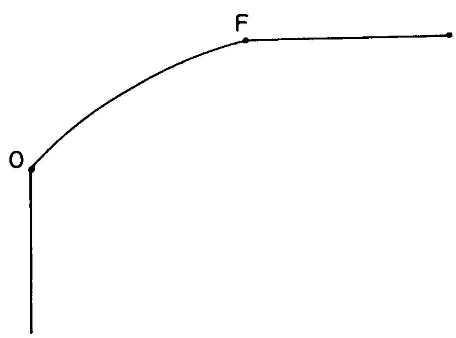

Figure 3. Curved element at the singularity

The total length of OF is

$$l = s(x_1) = (x_1^{2/3} + \tfrac{4}{9}a^2)^{3/2} - \tfrac{8}{27}a^3. \tag{18}$$

The functions ψ and $\partial\psi/\partial n$ are assumed to vary linearly on the curved element OF as

$$\psi \simeq (1 - s/l)\psi_{j-1} + (s/l)\psi_j, \quad \partial\psi/\partial n \simeq (1 - s/l)\psi_{j-1}^{(n)} + (s/l)\psi_j^{(n)}. \tag{19}$$

The system matrices **A** and **B** now include contributions from integration along the curved element. These, due to their complicated nature, can no longer be evaluated analytically and are therefore evaluated numerically by Gaussian quadrature.

NUMERICAL RESULTS

The direct boundary element method can now be applied to the problem. In order to compare the results with the results of Mogel and Street[4] and Aitchison,[5] the dimensions of the flow region are taken to be (Figure 2)

$$\mathrm{DE} = 1{\cdot}0, \quad \mathrm{EO} = 0{\cdot}1, \quad \mathrm{CB} = 1{\cdot}5, \quad \mathrm{CD} = 1{\cdot}0.$$

The elements on the free boundary are chosen so as to have equal length projections on the x axis. Runs with 5, 10, 15, 20 and 25 elements on the free boundary are performed for both the case when straight line segments are used everywhere and the case when a curved element is introduced at the singularity. Solution of the resulting system of non-linear equations produced the constant value of $\partial\psi/\partial n$ on the free boundary and the heights of the nodes there. Listing the heights of each node on the free boundary is impractical and so only one such value is examined; namely, the height of the last node at A, which is denoted by b.

Table I displays the values obtained for $\partial\psi/\partial n$ and b for different numbers of elements on the free boundary for both methods. From Table I it is observed that all the displayed quantities behave in the following way. If

$$n = 5k, \tag{20}$$

where n is the number of elements on the free boundary, it can easily be seen that, if f represents any of the above quantities,

$$f_{k+1} - f_k = \tfrac{1}{2}(f_k - f_{k-1}). \tag{21}$$

Subsequent extrapolation to the limit f^* leads to

$$f^* = 2f_5 - f_4. \tag{22}$$

Using equation (22), the results obtained after extrapolation are compared with the existing ones in Table II. The best free boundary obtained is displayed in Figure 4 (25 elements on free boundary, use of curved element).

Table I. Summary of values obtained for $\partial\psi/\partial n$ and b

Elements on free boundary	5	10	15	20	25
$\partial\psi/\partial n$ (BEM)	1·4283	1·4507	1·4600	1·4648	1·4673
$\partial\psi/\partial n$ (BEM with curved element)	1·4426	1·4590	1·4669	1·4711	1·4732
b (BEM)	0·2031	0·2118	0·2156	0·2176	0·2187
b (BEM with curved element)	0·2098	0·2158	0·2189	0·2206	0·2214

Table II. Comparison of results

Method	$\partial\psi/\partial n$	b
BEM	1·4698	0·2198
BEM with curved element	1·4753	0·2222
Aitchison	1·429	0·2013
Mogel and Street	1·562	0·245

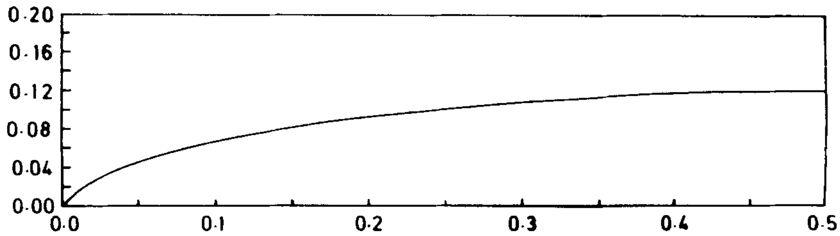

Figure 4. Free boundary for 25 elements, with a curved element at the singularity

CONCLUSION

In this study, the direct boundary element method is applied to a free boundary problem. This leads to the solution of a system of non-linear differential equations which generates the position of the free boundary and the values of the other unknown quantities of the problem. This implementation is improved by the incorporation of the analytical behaviour of the free boundary near the separation point. The improvement does not introduce further difficulties because of the properties of boundary element methods to deal only with the boundary of the region under consideration. The numerical results compare well with existing ones obtained by other numerical methods.

REFERENCES

1. T. Y. Wu, 'Inviscid cavity and wake flows', in M. Holt (ed.), *Basic Developments in Fluid Dynamics, Vol. 1*, Academic Press, New York, 1968.
2. G. Birkhoff and E. H. Zarantello, *Jets, Wakes and Cavities*, Academic Press, New York, 1957.
3. D. Riabouchinsky, 'On steady fluid motion with free surfaces', *Proc. London Math. Soc.*, **19**, 206 (1919).
4. T. R. Mogel and R. L. Street, 'A numerical method for steady state cavity flows', *J. Ship Res.*, **18**, 22 (1974).
5. J. M. Aitchison, 'The numerical solution of planar and axisymmetric cavitational flow problems', *Comput. Fluids*, **12** (1), 55 (1984).
6. C. Brennen, 'A numerical solution of axisymmetric cavity flows', *J. Fluid Mech.*, **37**, 671 (1969).
7. L. Fox and R. Sankar, 'The Regula–Falsi method for free boundary problems', *J. Inst. Math. Appl.*, **12**, 49 (1973).
8. J. A. Liggett, 'Location of free surface in porous media', *J. Hydr. Div. ASCE*, **103** (HY4), 353 (1977).
9. M. A. Kelmanson, 'Boundary integral equation solution of viscous flows with free surfaces', *J. Eng. Math.*, **17**, 329 (1983).
10. J. M. Aitchison and A. Karageorghis, 'Boundary integral techniques for free boundary problems', in A. Bossavit, A. Damlamian and M. Fremond (eds), *Free Boundary Problems: Applications and Theory, Vol. IV*, Pitman, London, 1985.
11. A. Karageorghis, 'Numerical solution of a shallow dam problem by a boundary element method', *Comput. Methods Appl. Mech. Eng.*, **61**, 265 (1987).
12. M. A. Jaswon and G. T. Symm, *Integral Equation Methods in Potential Theory and Elastostatics*, Academic Press, New York, 1977.
13. D. B. Ingham, P. J. Heggs and M. Manzoor, 'The numerical solution of plane potential problems by improved integral equation methods', *J. Comput. Phys.*, **42**, 77 (1981).
14. M. J. D. Powell, 'A hybrid method for nonlinear equations', in P. Rabinowitz (ed.), *Numerical Methods for Nonlinear Algebraic Equations*, Gordon and Breach, London, 1970.